Concrete Mathematics: Exploring Sums

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Outline

Abstract

In this paper, we explore various methodologies to calculate sums. We begin by looking at simple methods: converting sums into recurrences. We proceed to discuss common manipulations, such as using the repertoire method or perturbation. Finally, we conclude by extrapolating sums to both finite and infinite calculus mainly by building integrals.

Sums and Recurrences

The sum

$$S_n = \sum_{k=0}^{n} a_k$$

is the recurrence

$$S_0 = a_0$$

$$S_{n-1} + a_n$$

for $n > 0$

Sums and Recurrences Continued

Therefore, sums can be evaluated in closed form by solving recurrences. Assuming a_n equals a constant plus a multiple of n, the recurrence above can be rewritten as

$$R_0 = \alpha$$

$$R_n = R_{n-1} + \beta + \gamma n$$

for $n > 0$ Thus, we can generalize a solution such that:

$$R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

Manipulating Sums

Distributive Property

$$\sum_{k \in K} c a_k = c \sum_{k \in K} a_k$$

Associative Property

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$$

Commutative Property

$$\sum_{k \in K} (a_k) = \sum_{p(k) \in K} a_{p(k)}$$

Manipulating Sums Continued

Iverson Convention

From any logical statements made in middle of a formula, the values 0 or 1 can be obtained.

Multiple Sums Continued

Example

The following example demonstrates an important rule for combining different sets of indices. If K and K' are any sets of integers, then

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cup K'} a_k + \sum_{k \in K \cap K'} a_k$$

This is derived from the general formulas

$$\sum_{k \in K} a_k = \sum_{k} a_k [k \in K]$$

and

$$[k \in K] + [k \in K'] = [k \in K \cap K'] + [k \in K \cup K']$$

We use this method to either combine 2 almost disjoint index sets or to split off a single term from a set.

Manipulating Sums Continued

Perturbation

Start with an unknown sum and call it

$$S_n + a_{n+1} = \sum_{0 \le k \le n+1} a_k = a_0 + \sum_{1 \le k \le n+1} a_k$$

$$= a_0 + \sum_{1 \le k+1 \le n+1} a_{k+1}$$

$$= a_0 + \sum_{0 \le k \le n} a_{k+1}$$

This expression can then be simply expressed in terms of S_n

Manipulation Sums Perturbation

Example

$$S_n = \sum_{0 \le k \le n} k2^k$$

We then have

$$S_n + (n+1)2^{n+1} = \sum_{0 \le k \le n} (k+1)2^{(k+1)}$$

Using the associative property gives

$$\sum_{0 \le k \le n} k2^{(k+1)} + \sum_{0 \le k \le n} 2^{(k+1)}$$

Manipulation Sums Perturbation

Example

The first sum is simply $2S_n$ The second sum is just a geometric progression that yields $2^{n+2} - 2$ Therefore, the total sum becomes

$$(n-1)2^{n+1} + 2$$

Multiple Sums

Multiple Sums

The terms of a sum might be specified by two or more indices.

General Distributive Law

$$\sum_{j \in J, k \in K} a_j b_k = (\sum_{j \in J} a_j) + (\sum_{k \in K} b_k)$$

Multiple Sums Continued

Example

Evaluate

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j}$$

Replace k with $k+j$

$$= \sum_{1 \le j < k+j \le n} \frac{1}{k}$$

Summing on j, which is trivial, gives

$$= \sum_{1 \le k \le n} \sum 1 \le j \le n-k \frac{1}{k} = \sum_{1 \le k \le n} \frac{n-k}{k}$$

Multiple Sums Continued

Example

Then, by the associative law, we have

$$\sum_{1 \le k \le n} \frac{n}{k} - \sum_{1 \le k \le n} 1 = n\left(\sum_{1 \le k \le n} \frac{1}{k}\right) - n$$

This has the harmonic mean, and is a final answer for the original sum.

Finite Calculus

Difference Operator, Definite Sum, and Shift

$$\Delta f(x) = f(x+1) - f(x)$$

$$\sum_a^b g(x)\delta x = \sum_{a \le k < b} g(k)$$

$$Ef(x) = f(x+1)$$

Properties

Multiplicative Rule:

$$\Delta fg = f\Delta g + Eg\Delta f$$

Integration analogue

$$\sum_a^b g(x)\delta x = -\sum_b^a g(x)\delta x$$

$$\sum_a^b g(x)\delta x + \sum_b^c g(x)\delta x = \sum_a^c g(x)\delta x$$

Properties (cont.)

Fundamental Theorem of (Finite) Calculus:

$$g(x) = \Delta f(x) \Rightarrow \sum_{a}^{b} g(x) = f(b) - f(a)$$

Chain Rule:

$$\sum u\Delta v = uv - \sum Ev\Delta u$$

Finite Calculus Analogues: Rising and Falling Powers

Definition

Rising and Falling Powers

$$\forall m > 0 : x^{\underline{m}} = x(x-1)(x-2)...(x-m+1)$$

$$\forall m > 0 : x^{\overline{m}} = x(x+1)(x+2)...(x+m-1)$$

$$x^{\underline{0}} = x^{\overline{0}} = 1$$

$$\forall m > 0 : x^{\underline{-m}} = \frac{1}{(x+1)(x+2)...(x+m)}$$

$$\forall m > 0 : x^{\overline{-m}} = \frac{1}{(x-1)(x-2)...(x-m)}$$

Finite Calculus Analogues: Rising and Falling Powers

Properties of Rising and Falling Powers

$$\Delta x^{\underline{m}} = m x^{\underline{m-1}}$$

$$\sum_{0 \le k < n} k^{\underline{m}} = \frac{n^{\underline{m+1}}}{m+1}$$

$$\sum_{a \le k < b} c^k = \frac{c^b - c^a}{c - 1}$$

Infinite Series

Definition

Infinite Series: Suppose $a_k \geq 0 \forall k \in K$. If there is a number A such that for any finite $F \subset K$:

$$\sum_{k \in F} a_k \leq A$$

Then $\sum_{k \in K} a_k$ exists and is equal to the least such A.

Infinite Series

Generalization for any (multi-dimensional) index set K and any sequence a_k:

$$A = \sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-$$

where:

$$a_k^+ = a_k.[a_k > 0]$$
$$a_k^- = a_k.[a_k < 0]$$

Infinite Series

We have the following scenarios:

- A^+ and A^- are finite: A converges absolutely to $A^+ + A^-$.
- A^+ is infinite, A^- is finite: A diverges to ∞.
- A^- is infinite, A^+ is finite: then A diverges to $-\infty$.
- A^+ and A^- are infinite: A is undefined.

Infinite Series

Fundamental Principle of Multiple Sums

Absolutely convergent sums over 2 or more indices can always be summed first with respect to any one of those indices.

Infinite Series

Formally, if J and the elements of $\{K_j | j \in J\}$ are sets of indices such that:

$$\sum_{j \in J, k \in K_j} a_{j,k} \text{ converges absolutely to } A$$

Then there exists A_j for each $j \in J$ such that

$$\sum_{k \in K_j} a_{j,k} \text{ converges absolutely to } A_j$$

$$\sum_{j \in J} A_j \text{ converges absolutely to } A$$